

The energy-dependent $\pi^+\pi^+\pi^+$ scattering amplitude from QCD

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Focusing on three-pion states with maximal isospin ($\pi^+\pi^+\pi^+$), we present the first non-perturbative determination of an energy-dependent three-hadron scattering amplitude from first-principles QCD. The calculation combines finite-volume three-hadron energies, extracted using numerical lattice QCD, with a relativistic finite-volume formalism, required to interpret the results. To fully implement the latter, we also solve integral equations that relate an intermediate three-body K matrix to the physical three-hadron scattering amplitude. The resulting amplitude shows rich analytic structure and a complicated dependence on the two-pion invariant masses, represented here via Dalitz-like plots of the scattering rate.

Introduction — The three-body problem lies at the core of a broad range of outstanding questions in quantum chromodynamics (QCD). The largest uncertainty in QCD-based structure calculations of light nuclei, for example, is the estimate of the three-nucleon force (see Ref. [1]). In addition, many QCD resonances have significant branching fraction to channels with three or more hadrons. The Roper resonance, for example, has defied simple quark-model descriptions, due in part to its nature as a broad resonance with a $\sim 30\%$ branching fraction to $N\pi\pi$. A rigorous QCD calculation would elucidate the role of non-perturbative dynamics in the Roper’s peculiar properties, e.g. the fact that it has a lower mass than the negative-parity ground state, which seems unnatural from the perspective of the quark model [2, 3].

As a necessary step towards studying a broad class of three-hadron systems, in this work we present the first study of an energy-dependent three-body scattering amplitude from QCD. This non-perturbative result is achieved by the coalescence of three novel techniques: a calculation of finite-volume three-hadron energies based in numerical lattice QCD, a relativistic finite-volume formalism to relate the energies to K matrices, and a numerical evaluation of corresponding integral equations to convert the latter into the three-hadron scattering amplitude. The theoretical basis required to achieve these final two steps was derived in Refs. [4, 5].¹

This work considers the scattering of three-pion states with maximal isospin ($I = 3$) in QCD with three dynamical quarks ($N_f = 2 + 1$): two degenerate light quarks,

with heavier-than-physical mass corresponding to a pion mass $m_\pi \approx 391$ MeV, and a strange quark. This channel offers an optimal benchmark case, since both the maximal-isospin three-pion system and its two-pion subsystem are expected to be weakly interacting and non-resonant.

Many numerical studies of three-hadron states have been published over the last decade, ranging from early work deriving and fitting large-volume expansions of the three-pion ground state [46–48] to more recent results using quantization conditions to study ground [49] and excited states [50–52], with the latter set each analyzing the lattice QCD spectrum published in Ref. [53]. Independent sets of finite-volume energies have also been calculated and analyzed in Refs. [54] and [55]. The present investigation goes beyond this previous work, by providing the first complete numerical determination of physical scattering amplitudes for three-body systems.

In the following, we first discuss our determination of two- and three-pion finite-volume energies, before describing the fits used to relate these to infinite-volume K matrices. The latter then serve as inputs to known integral equations, which we solve numerically to extract the $3\pi^+ \rightarrow 3\pi^+$ scattering amplitude. Additional details of the analysis are discussed in the supplementary material.

Spectral Determination — Figure 1 summarizes the two- and three-pion finite-volume spectra calculated in this work.²

Computations were performed on anisotropic lattices which have a temporal lattice spacing, a_t , finer than the spatial lattice spacing, a_s ($a_t = a_s/\xi$ with $\xi = 3.444(6)$ [56]). Two lattice ensembles were used, differing only in the lattice geometry with the smaller spatial volume characterized by $20^3 \times 256$ and the larger by $24^3 \times 128$. We use $2 + 1$ flavors of dynamical clover

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¹ A large body of work has investigated general methods for relating finite-volume energies to scattering amplitudes for both two- and three-body states. See Refs. [6–24] and Refs. [25–45], respectively.

² Two-pion energies on the larger volume have already appeared in Ref. [56].

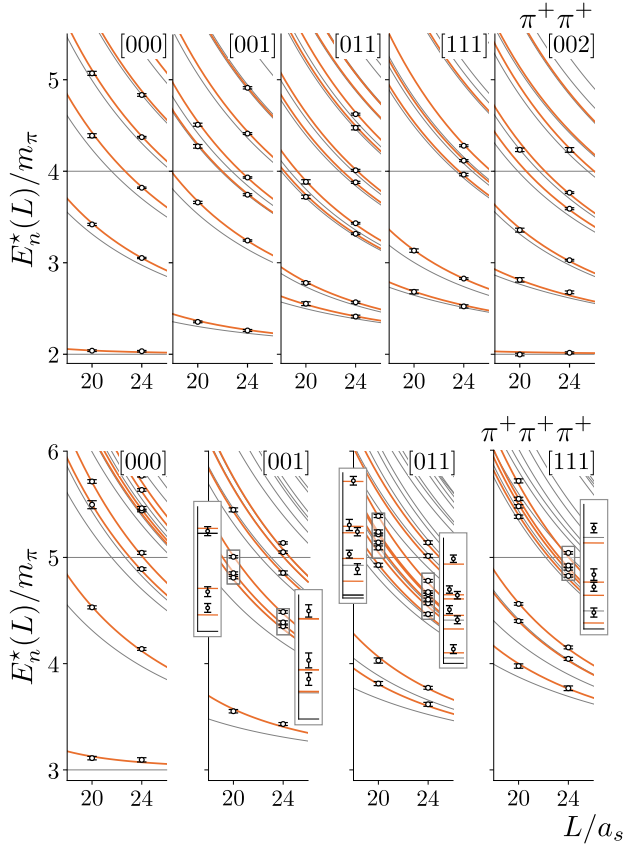


FIG. 1. The $\pi^+\pi^+$ and $\pi^+\pi^+\pi^+$ finite-volume spectra in the center-of-momentum frame for the relevant finite-volume irreps with various overall momenta, as explained in the text. Points are computed energy levels on the two volumes with error bars showing statistical uncertainties. Each rectangular insert shows a vertical zoom of the region indicated by the small neighboring rectangle. Grey curves are the “non-interacting” finite-volume energies, i.e. the energies in the absence of any interactions between pions. Orange curves are predictions from the finite-volume formalism based only on the two-particle scattering length, given in Eq. (4) (here with the local three-body interaction set to zero).

fermions, with three-dimensional stout-link smearing in the fermion action, and a tree-level Symanzik-improved gauge action. The bare parameters and basic lattice properties are detailed in Refs. [57, 58]. Setting the scale via $a_t^{-1} = m_\Omega^{\text{exp}} (a_t m_\Omega^{\text{latt}})^{-1}$,³ and combining with $a_t m_\pi = 0.06906(13)$ [56] and $a_t m_K = 0.09698(9)$ [61], yields $m_\pi \approx 391$ MeV and $m_K \approx 550$ MeV. The values of $a_t m_\pi$ and ξ translate into spatial extents of $m_\pi L = 4.76$ and $m_\pi L = 5.71$ for the two ensembles.

The spectrum of energies in a finite volume is discrete and each energy level provides a constraint on the scattering amplitudes at the corresponding center-of-momentum energy. To obtain more constraints, we also include spec-

tra for which the systems have overall non-zero momentum, \mathbf{P} . Momenta are quantized by the cubic spatial boundary conditions, $\mathbf{P} = \frac{2\pi}{L}(n_1, n_2, n_3)$, where $\{n_i\}$ are integers, and we write this using a shorthand notation as $[n_1 n_2 n_3]$.

In this work we restrict attention to S -wave scattering. The reduced symmetry of a cubic lattice means that total angular momentum, J , is not a good quantum number and instead channels are labelled by the irreducible representation (irrep, Λ) of the octahedral group with parity for $\mathbf{P} = \mathbf{0}$ or the relevant subgroup that leaves \mathbf{P} invariant for $\mathbf{P} \neq \mathbf{0}$ [62, 63]. We consider the relevant irreps which contain $J = 0$: $A_1^-(A_1^+)$ for $\pi\pi\pi$ ($\pi\pi$) at rest and $A_2(A_1)$ for $\pi\pi\pi$ ($\pi\pi$) with non-zero \mathbf{P} . Isospin, I , and G -parity, G , are good quantum numbers in our lattice formulation; these distinguish the two-pion ($I^G = 2^+$) and three-pion ($I^G = 3^-$) channels. We neglect higher partial waves here, in particular the two-particle D -wave which mixes with the S -wave in the finite-volume energies. As described in Ref. [56], a nonzero D -wave interaction can be extracted, in particular if aided by the consideration of other, non-trivial finite-volume irreps.⁴

To reliably extract the finite-volume energies we have computed two-point correlation functions using a large basis of appropriate interpolating operators. From these, the spectra are determined using the variational method [64–66], with our implementation described in Refs. [67, 68]. This amounts to calculating a matrix of correlation functions,

$$G_{ij}(t) = \langle \mathcal{O}_i(t) \mathcal{O}_j^\dagger(0) \rangle, \quad (1)$$

and diagonalizing $M(t, t_0) = G(t_0)^{-1/2} \cdot G(t) \cdot G(t_0)^{-1/2}$ for a fixed t_0 . One can show that the corresponding eigenvalues satisfy $\lambda_n(t, t_0) \rightarrow e^{-E_n(L)(t-t_0)}$, where $E_n(L)$ is the n ’th energy level with overlap to some of the operators in the basis. This basic methodology has been applied to a wide range of two-hadron scattering observables for several phenomenologically interesting channels [56, 69–80]. See Sec. 1 of the supplementary material for some example plots of $\lambda_n(t, t_0)$.

In order to robustly interpolate the two- and three-pion energy eigenstates we use operators with two- and three-meson-like structures in the appropriate irrep, constructed from products of single-meson-like operators projected to definite spatial momentum. The latter are built from linear combinations, chosen to optimize overlap to the single-pion states, of fermion bilinears of the form, $\bar{\psi} \Gamma D \dots D \psi$, where ψ is a quark field and D is a discretized covariant derivative. Details of these operator constructions are given in Sec. 5 of the supplementary material. Using

³ where $a_t m_\Omega^{\text{latt}} = 0.2951(22)$ was measured in Ref. [59] and m_Ω^{exp} is the experimentally determined Ω baryon mass from Ref. [60]

⁴ There is, in principle, a systematic uncertainty associated with neglecting the D -wave contribution. Given the consistency of our results with Ref. [56], this appears to be below the statistical uncertainty in the present fits. See also Secs. VIII A and B of that work for more discussion.

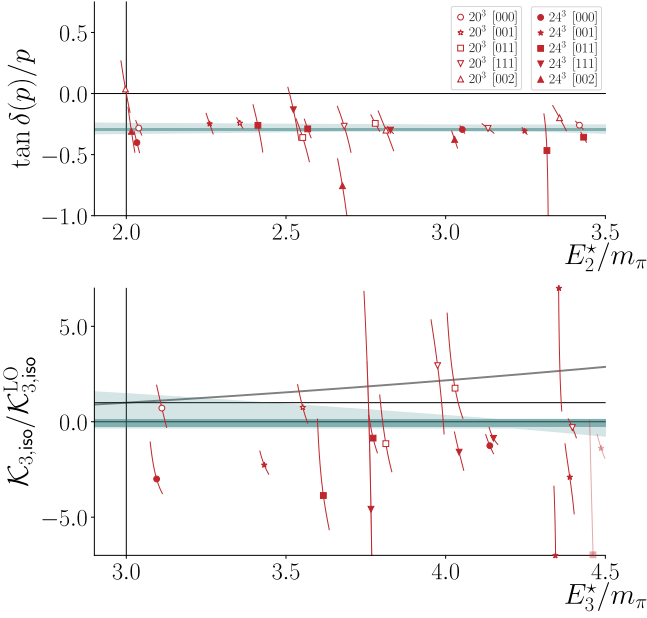


FIG. 2. Example of data and fits for \mathcal{K}_2 and $\mathcal{K}_{3,\text{iso}}$, as described in the text. The red points are given by substituting finite-volume energies into $-1/F(E_2, \mathbf{P}, L)$ and $-1/F_{3,\text{iso}}(E_3, \mathbf{P}, L)$ for the two- and three-particle energies, respectively, with the volume and \mathbf{P} indicated in the legend. A symbol appearing at the very top or bottom represents a case where the central value falls outside the plotted region. The dark cyan bands represent the fit shown in Eq. (4) while the lighter bands show the spread covered by the various fits described in the supplemental material. For the bottom panel we normalize to $m_\pi^2 \mathcal{K}_{3,\text{iso}}^{\text{LO}} = 4608 \pi^2 (m_\pi a_0)^2$, with $m_\pi a_0$ taken from Eq. (4). This simple relation between $\mathcal{K}_{3,\text{iso}}$ and the two-particle scattering length holds at leading order in chiral perturbation theory at threshold, as was first derived in Ref. [50]. The grey curve gives the full leading-order prediction, which is linear in E_3^2 .

such a wide variety of optimized operators, and especially multi-hadron operators with momentum-projected single-hadron components, allows one to minimize excited state contamination and extract the energies reliably and precisely from small values of t . This approach is made feasible due to the distillation method [81] which we employ to efficiently compute the numerous quark-field Wick contractions that are required.

Returning to the two- and three-pion spectra summarized in Fig. 1, we observe a one-to-one correspondence between the computed energy levels and the non-interacting energies in all panels, with the computed values slightly higher in energy than the non-interacting levels. This suggests that the system is weakly interacting and repulsive in both the two- and three-hadron sectors.

Analyzing the finite-volume spectra — We now describe our method for determining two- and three-body K matrices from the extracted finite-volume energies, beginning with an overview of scattering observables:

The two-pion scattering amplitude is defined as the

connected part of the overlap between an incoming $\pi^+\pi^+$ asymptotic state (with momenta $\mathbf{p}, -\mathbf{p}$) to an outgoing $\pi^+\pi^+$ state (with $\mathbf{p}', -\mathbf{p}'$). Without loss of generality, here we have assumed the center-of-momentum frame. We also define $p = |\mathbf{p}| = |\mathbf{p}'|$, where we have used that the magnitudes must be equal to satisfy energy conservation. In addition, $s_2 \equiv E_2^2 \equiv 4(p^2 + m_\pi^2)$ defines the squared center-of-momentum frame energy. The only remaining degree of freedom is the scattering angle between \mathbf{p} and \mathbf{p}' . In this work we focus on the S -wave scattering amplitude, denoted \mathcal{M}_2 , in which this angle is integrated to project onto zero-angular-momentum states. Finally we recall the simple relation between \mathcal{M}_2 and the K matrix in the elastic region, $\mathcal{K}_2^{-1} = \text{Re } \mathcal{M}_2^{-1}$.⁵ In contrast to \mathcal{M}_2 , \mathcal{K}_2 is real for real s_2 and is meromorphic in a region of the complex s_2 plane around $s_2 = 4m_\pi^2$. In this work we also consider an analogous, three-body K matrix, introduced in Ref. [4] and denoted by $\mathcal{K}_{\text{df},3}$.

In the two-pion sector, in the case that the S -wave interactions are dominant, the scalar-irrep finite-volume energies satisfy the quantization condition [6, 7, 9],

$$\mathcal{K}_2(E_2^*) + F^{-1}(E_2, \mathbf{P}, L) = 0, \quad (2)$$

where $E_2^* \equiv \sqrt{E_2^2 - \mathbf{P}^2}$ is the center-of-momentum energy and $F(E_2, \mathbf{P}, L)$ is a known geometric function. For the three-body sector, we use the isotropic approximation of the general formalism derived in Ref [4], which takes an analogous form, now for pseudoscalar-irrep energies

$$\mathcal{K}_{3,\text{iso}}(E_3^*) + F_{3,\text{iso}}^{-1}[\mathcal{K}_2](E_3, \mathbf{P}, L) = 0, \quad (3)$$

where the notation is meant to stress that $F_{3,\text{iso}}[\mathcal{K}_2](E_3, \mathbf{P}, L)$ is a functional of $\mathcal{K}_2(E_2^*)$. $F_{3,\text{iso}}$ is defined in Eq. (39) of Ref. [4]. Here $\mathcal{K}_{3,\text{iso}}$ is the component of $\mathcal{K}_{\text{df},3}$ that only depends on the total three-hadron energy, i.e. is “isotropic”. Equation (3) holds only when $\mathcal{K}_{\text{df},3}$ is well approximated to be isotropic and our fits give evidence that this is a good approximation for this system.

Combining these two conditions with the energies plotted in Fig. 1 allows one to constrain both the two- and three-hadron K matrices. One strategy is to fit a parametrization of \mathcal{K}_2 and use this to determine the energy dependence of $\mathcal{K}_{3,\text{iso}}$ as summarized in Fig. 2. An alternative approach is to parametrize both K matrices and fit these simultaneously to the entire set of finite-volume energies. A detailed discussion with a wide range of fits is given in Sec. 2 of the supplementary material. Both strategies give consistent results and the key message is that the full data set is well described by a constant $\mathcal{K}_{3,\text{iso}}$ that is consistent with zero, together with the leading-order effective range expansion: $\tan \delta(p) = -a_0 p$ with

⁵ The imaginary part of \mathcal{M}_2^{-1} is completely fixed by unitarity so that \mathcal{K}_2 is the only part free to depend on the dynamics of the system. We work with the simple phase space factor, proportional to the momentum magnitude. See, e.g., Ref. [23] for more details.

$\mathcal{K}_2(E_2^*) = -16\pi E_2^* \tan \delta(p)/p$. Here the second equation defines the S -wave scattering phase shift, $\delta(p)$, and the first defines the scattering length, a_0 . Our best fit, performed simultaneously to all spectra shown in Fig. 1 but with a cutoff in the center-of-momentum frame energies included,⁶ yields

$$\begin{aligned} m_\pi a_0 &= 0.296 \pm 0.008 \begin{bmatrix} 1.0 & 0.6 \\ & 1.0 \end{bmatrix}, \\ m_\pi^2 \mathcal{K}_{3,\text{iso}} &= -339 \pm 770 \end{aligned} \quad (4)$$

with a χ^2 per degree-of-freedom of $64.5/(37 - 2) = 1.84$. The square-bracketed matrix gives the correlation between the two fit parameters. This is consistent with the previous determination of the scattering length at this pion mass, presented in Ref. [56], and is also the value used to generate the orange curves in Fig. 1 (together with $\mathcal{K}_{3,\text{iso}} = 0$). In Fig. 2 we illustrate the same fit using the darker cyan curves. In addition, we include the lighter bands as a systematic uncertainty, estimated from the spread of various constant and linear fits, as detailed in Sec. 2 of the supplementary material.

$3\pi^+$ scattering amplitude — Following the relativistic integral equations presented in Ref. [5], we can write the $J = 0$ and $\mathcal{K}_{3,\text{iso}} = 0$ amplitude as follows:

$$\begin{aligned} \mathcal{M}_3^{(u,u)}(p, k) &= -\mathcal{M}_2(E_{2,p}^*) G_s(p, k) \mathcal{M}_2(E_{2,k}^*) \\ &\quad - \mathcal{M}_2(E_{2,p}^*) \int_{k'} G_s(p, k') \mathcal{M}_3^{(u,u)}(k', k), \end{aligned} \quad (5)$$

where $\int_k \equiv \int dk k^2 / [(2\pi)^2 \omega_k]$ and we have introduced

$$G_s(p, k) \equiv -\frac{H(p, k)}{4pk} \log \left[\frac{\alpha(p, k) - 2pk + i\epsilon}{\alpha(p, k) + 2pk + i\epsilon} \right], \quad (6)$$

$$\alpha(p, k) \equiv (E_3 - \omega_k - \omega_p)^2 - p^2 - k^2 - m^2. \quad (7)$$

\mathcal{M}_2 is the S -wave two-particle scattering amplitude, introduced above, which depends on the invariant $E_{2,k}^{*2} \equiv (E_3 - \omega_k)^2 - k^2$, with $\omega_k = \sqrt{k^2 + m^2}$. The (u, u) superscript emphasizes that specific spectator momenta, k and p , are singled out in the initial and final states respectively. The function G_s encodes the spectator exchange, projected to the S -wave. It inherits a scheme dependence through the smooth cutoff function H , defined in Eqs. (28) and (29) of Ref. [4]. This scheme dependence is matched by that inside of $\mathcal{K}_{3,\text{iso}}$ such that the resulting scattering amplitude is universal.

To use Eq. (5) in practice, one requires a parameterization for \mathcal{M}_2 . As described in the previous section, the

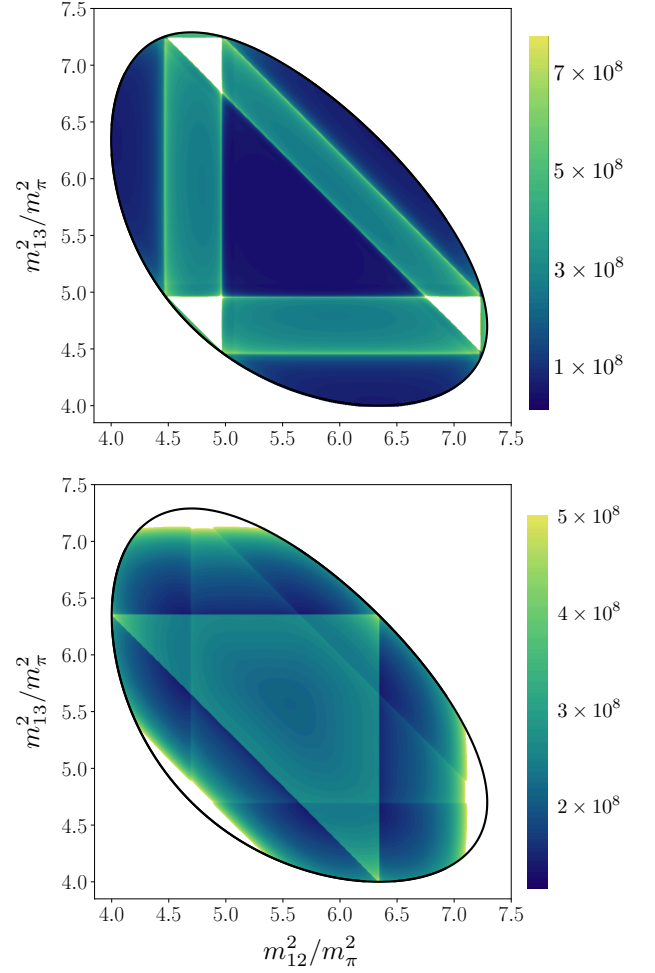


FIG. 3. *Top*: Dalitz-like plot of $m_\pi^4 |\mathcal{M}_3|^2$ for $\sqrt{s_3} = 3.7m$ with final kinematics fixed to $\{\mathbf{p}_1^2, \mathbf{p}_2^2\} = \{0.01m_\pi^2, 0.7m_\pi^2\} \Rightarrow \{m_{12}', m_{13}'\} = \{2.1m_\pi, 2.25m_\pi\}$. *Bottom*: Same total energy, now with incoming and outgoing kinematics set equal, as discussed in the text.

$\pi^+\pi^+$ system is well described using the leading order effective range expansion for \mathcal{M}_2 ,

$$\mathcal{M}_2(E_2^*) = \frac{16\pi E_2^*}{-1/a_0 - i\sqrt{E_2^{*2}/4 - m_\pi^2}}. \quad (8)$$

Following the derivation of Ref. [5], the final step is to symmetrize with respect to the spectators, to reach

$$\mathcal{M}_3(s_3, m_{12}^{\prime 2}, m_{13}^{\prime 2}, m_{12}^2, m_{13}^2) = \sum_{\mathbf{p}_i \in \mathcal{P}_p} \sum_{\mathbf{k} \in \mathcal{P}_k} \mathcal{M}_3^{(u,u)}(p, k), \quad (9)$$

where $\mathcal{P}_p = \{\mathbf{p}, \mathbf{a}', -\mathbf{p} - \mathbf{a}'\}$ and $\mathcal{P}_k = \{\mathbf{k}, \mathbf{a}, -\mathbf{k} - \mathbf{a}\}$. We have presented the left-hand side as a function of the five Lorentz invariants that survive after truncating to $J = 0$ in both the two and three particle sector: the squared three-hadron center-of-momentum frame energy, s_3 , as well as two pion-pair invariant masses for each of the initial and final states. These are defined by introducing the

⁶ This fit is denoted by B₂₊₃ in Sec. 2 of the supplementary material. As explained there, the fitted data includes all two-pion energies below $E_{2,\text{cut}}^* = 3.4m_\pi$ and all three-pion energies below $E_{3,\text{cut}}^* = 4.4m_\pi$, with both cutoffs applied to energies in the center-of-momentum frame.

notation $\{\mathbf{k}, \mathbf{a}, -\mathbf{k} - \mathbf{a}\} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$, then for example

$$m_{12}^2 = (p_1 + p_2)^2 = (E_3^* - [m_\pi^2 + \mathbf{p}_3^2]^{1/2}) - \mathbf{p}_3^2, \quad (10)$$

where the middle expression depends on on-shell four-vectors with $p_1^2 = m_\pi^2$.

In the top panel of Fig. 3 we show a Dalitz-like plot of $|\mathcal{M}_3|^2$ as a function of (m_{12}, m_{13}) , with all other kinematics fixed as indicated in the caption. In a usual Dalitz description, the incoming energy is fixed by the decaying particle so that only the outgoing kinematics can vary, whereas here we simply fix the other kinematics. The inputs to this plot are the best-fit scattering length, given in Eq. (4), together with $\mathcal{K}_{3,\text{iso}} = 0$. The bottom panel of Fig. 3 shows the same $\sqrt{s_3}$ but varies incoming and outgoing kinematics according to $m_{12} = m'_{12}$ and $m_{13} = m'_{13}$.

Additional details concerning the S -wave integral equations are presented in Secs. 3 and 4 of the supplementary material, where we also describe the propagation of the uncertainties of $m_\pi a_0$ and $\mathcal{K}_{3,\text{iso}}$ into the predicted amplitude.⁷

Summary — In this work we have presented the first lattice QCD determination of the energy-dependent three-to-three scattering amplitude for three pions with maximal isospin. The calculation proceeded in three steps: (i) determining finite-volume energies with $\pi^+\pi^+\pi^+$ quantum numbers, (ii) using the framework of Ref. [4] to extract two- and three-body K matrices from these, and (iii) applying the results of Ref. [5] to convert these to the three-hadron scattering amplitude, by solving known integral equations. The three steps are summarized, respectively, by Figs. 1, 2 and 3 of the text.

Having established this general workflow, it is now well within reach to rigorously extract three-hadron resonance properties from lattice QCD calculations. In particular the formalism has recently been extended to three-pion states with any value of isospin in Ref. [42]. This should enable studies, for example, of the ω , h_1 and a_1 resonances. The main outstanding challenges here include rigorous resonant parametrizations of the intermediate three-body K matrix, as well as a better understanding of the analytic continuation required to identify the resonance pole position.

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⁷ See Ref. [82] for a discussion of integral equations and their solutions in a resonant three-hadron channel.

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